

## Math 54 - Practice Final Exam

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Here are some problems to help you prepare for the final exam. Note that there are 20 problems here — 10 linear algebra and 10 differential equations. Recall that the final will only have 8 problems (4 of each) so this practice final is much much longer than the actual final will be, but you should still be able to do all the problems listed here.

1. (a) Find the inverse of the following matrix.

$$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & -2 \\ -3 & 12 & -32 \end{pmatrix}$$

- (b) Solve the following system using the inverse found above.

$$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & -2 \\ -3 & 12 & -32 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$$

2. Let  $W$  be the set of all invertible  $n \times n$  matrices. Is  $W$  a subspace of  $M_{nn}$ ?  
3. Determine if the polynomials  $x + 1$ ,  $2x^2 + 3$ , and  $3x - 5$  are linearly independent. Do they span  $P_2$ ?  
4. Find bases for the null space, row space, and column space of the following matrix.

$$\begin{pmatrix} 1 & 1 & -3 & 3 & 5 \\ 4 & 4 & -14 & 12 & 22 \\ -3 & -3 & 5 & -6 & -5 \\ 1 & 1 & 3 & 6 & 5 \end{pmatrix}$$

5. (a) Let  $X = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ . Let  $T : M_{22} \rightarrow M_{22}$  be the function defined by  $T(A) = AX - XA$ . Prove that  $T$  is a linear transformation.

(b) Find a basis for the null space of  $T$ . (The null space of  $T$  is defined as the set of all matrices  $A$  such that  $T(A) = 0$ )

6. Let  $V$  be an inner product space. Using the Cauchy-Schwartz inequality, prove that for any vectors  $\mathbf{u}, \mathbf{v} \in V$ ,  $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ .

7. Let  $\mathbf{v}_1, \dots, \mathbf{v}_n$  be an orthonormal basis for an inner product space  $V$ . Prove that the coordinate vector relative to this basis of a vector  $\mathbf{x} \in V$  is

$$[\mathbf{x}] = \begin{pmatrix} \mathbf{x} \cdot \mathbf{v}_1 \\ \vdots \\ \mathbf{x} \cdot \mathbf{v}_n \end{pmatrix}.$$

8. (a) Find the determinant of the following matrix.

$$\begin{pmatrix} -2 & 3 & 2 \\ -1 & 3 & 0 \\ 4 & -3 & 1 \end{pmatrix}$$

(b) Is the above matrix invertible?

9. Find bases for the eigenspaces of the following matrix.

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

10. Determine if the following matrices are diagonalizable. If so, diagonalize them.

$$(a) \begin{pmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix} \quad (b) \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

11. (a) Prove that if  $\xi e^{rt}$  is a solution of a system of differential equations  $\mathbf{x}' = A\mathbf{x}$ , then  $r$  is an eigenvalue of  $A$  and  $\xi$  is an associated eigenvector.

(b) Solve the following system and draw its phase portrait.

$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

12. Consider the following system of linear differential equations.

$$\mathbf{x}' = \begin{pmatrix} 6 & 5 \\ 2 & -3 \end{pmatrix} \mathbf{x}$$

(a) Find the special fundamental matrix  $\Phi(t)$  which satisfies  $\Phi(0) = I$ .

(b) Solve the following initial value problem using the fundamental matrix found in

(a).

$$\mathbf{x}' = \begin{pmatrix} 6 & 5 \\ 2 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

(c) Draw the phase portrait of the given system.

13. (a) Prove that if  $A = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$  is diagonal, then  $e^{At} = \begin{pmatrix} e^{d_1 t} & & \\ & \ddots & \\ & & e^{d_n t} \end{pmatrix}$ .

(b) Prove that if  $A$  can be diagonalized as  $A = S\Lambda S^{-1}$ , then  $e^{At} = S e^{\Lambda t} S^{-1}$ .

(c) Compute  $e^{At}$  using (b) where  $A$  is the matrix in problem 11b.

14. (a) Prove that if  $\mathbf{u} + i\mathbf{v}$  is a complex solution of the system  $\mathbf{x}' = A\mathbf{x}$ , then both  $\mathbf{u}$  and  $\mathbf{v}$  are real solutions.

(b) Solve the following system.

$$\mathbf{x}' = \begin{pmatrix} 5 & 10 \\ -2 & -3 \end{pmatrix} \mathbf{x}$$

15. (a) Prove that if  $\mathbf{x} = \xi t e^{rt} + \eta e^{rt}$  is a solution of the system  $\mathbf{x}' = A\mathbf{x}$ , then  $r$  is an eigenvalue of  $A$ ,  $\xi$  is an associated eigenvector, and  $\eta$  satisfies  $(A - rI)\eta = \xi$ .

(b) Solve the following system.

$$\mathbf{x}' = \begin{pmatrix} -3 & -2 \\ 2 & -7 \end{pmatrix} \mathbf{x}$$

16. Find the eigenvalues and eigenfunctions of the following boundary value problem.

$$y'' + \lambda y = 0, \quad y(L) = 0, \quad y'(0) = 0$$

17. (a) Derive the formulas for the Fourier coefficients of a function with period  $2L$ .

(b) Find the Fourier series of the following function and draw the graph of the function to which the Fourier series converges for three periods.

$$f(x) = \begin{cases} 0, & -2 \leq x < 0 \\ 1 - x, & 0 \leq x < 2 \end{cases}$$

18. Let  $f(x) = 1 + x$  for  $0 < x < 1$ .

(a) Find the Fourier sine series of the given function.

(b) Find the Fourier cosine series of the given function.

19. Solve the following heat equation (from scratch).

$$4u_{xx} = u_t, \quad 0 < x < 2$$

with boundary conditions

$$u(0, t) = 0 = u(2, t)$$

and initial condition  $u(x, 0) = 2 \sin \pi x - 5 \sin 4\pi x$ .

20. Solve the following partial differential equation.

$$tu_{xx} = u_t, \quad 0 < x < \pi$$

with boundary conditions

$$u(0, t) = 0 = u(\pi, t)$$

and initial condition  $u(x, 0) = x$ .